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BPS R-balls in $\mathcal{N}=4$ SYM on $R \times S^3$, quantum Hall analogy and AdS/CFT holography

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Abstract: In this paper, we propose a new approach to study the BPS dynamics in $\mathcal{N}=4$ supersymmetric U(N) Yang-Mills theory on $R \times S^3$, in order to better understand the emergence of gravity in the gauge theory. Our approach is based on supersymmetric, space-filling Q-balls with R-charge, which we call R-balls. The usual collective coordinate method for non-topological scalar solitons is applied to quantize the half and quarter BPS R-balls. In each case, a different quantization method is also applied to confirm the results from the collective coordinate quantization. For finite N, the half BPS R-balls with a U(1) R-charge have a moduli space which, upon quantization, results in the states of a quantum Hall droplet with filling factor $\nu = 1$. These states are known to correspond to the "sources" in the Lin-Lunin-Maldacena geometries in IIB supergravity. For large N, we find a new class of quarter BPS R-balls with a non-commutativity parameter. Quantization on the moduli space of such R-balls gives rise to a non-commutative Chern-Simons matrix mechanics, which is known to describe a fractional quantum Hall system. In view of AdS/CFT holography, this demonstrates a profound connection of emergent quantum gravity with non-commutative geometry, of which the quantum Hall effect is a special case.

Keywords: AdS-CFT Correspondence, Supersymmetric gauge theory.

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1. Introduction

The AdS/CFT correspondence [1] is a gravity/gauge theory duality, according to which gravity is an emergent phenomenon in the dual gauge field theory. In the most well-understood case, the classical $AdS_5 \times S^5$ geometry is conjectured to be encoded in the strong 't Hooft coupling regime of the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on $R \times S^3$ with gauge group U(N). A thorough understanding of how this correspondence happens remains a challenge.

In the last two years, there has been encouraging progress. In ref. [2], Berenstein first proposed to consider a decoupled limit which singles out the half BPS sector in the $\mathcal{N} = 4$ SYM. As a model for the dynamics of the half BPS states, he studied the gauged mechanics of a holomorphic normal matrix, which was shown to be equivalent to a one dimensional free fermion system in phase space. (Similar results had been obtained in a complex matrix model [3].) The 1-d free fermion system can be mapped to an integer quantum Hall (IQH) droplet in two dimensions. An amazing and profound connection of the IQH droplet picture with type IIB geometries on the gravity side was subsequently revealed in a seminar paper by Lin, Lunin and Maldacena (LLM) [4]. By solving the equations of motion in IIB supergravity in ten dimensions, they have been able to obtain all non-singular half BPS type IIB geometries with isometry $R \times \mathrm{SO}(4) \times \mathrm{SO}(4)$, which turn out to be completely determined by the boundary value of a single real function z on a plane. The boundary value of z can be only $\pm 1/2$, which may be interpreted as the distributions of two types of point charge sources on the boundary plane. Either region with $z = 1/2$ or $z = -1/2$ can be viewed as a droplet of an incompressible fluid, like the IQH fluid. The simplest case is the familiar geometry $AdS_5 \times S^5$, which corresponds to a circular droplet on the boundary plane. In this way, one is tempted to associate the half BPS geometries in IIB supergravity with the half BPS states in $\mathcal{N} = 4$ SYM, by comparing the boundary IQH droplets in LLM's half BPS geometry to those in the phase space of Berenstein fermions. This comparison is justified only when one can make sense to Berenstein fermions in the half BPS sectors in $\mathcal{N} = 4$ SYM, not merely in a plausible model. Attempts to substantiate this comparison have been made in several recent papers [5, 6] with various degrees of success. (For the recent generalization of gauged matrix mechanics to 1/4 and 1/8 BPS states in $\mathcal{N} = 4$ SYM, see [7].)

In this paper we will directly attack the problem of how to see the Berenstein fermions emerging in the $\mathcal{N} = 4$ SYM by examining the BPS scalar field backgrounds. Our motivation came from the recognition that the emergence of gravity in SYM, as the correspondence between Berenstein and LLM fermions should imply, actually indicates the background independence on both sides of the holographic duality. To test the background independence, it is certainly desirable to have a systematic approach for finding candidate states on the gauge theory side for possible (classical or quantum) geometries on the gravity side. The simplest candidate states in SYM are the quantum BPS states with sufficiently large R-charge. For their explicit construction, we propose a new approach, which starts with constructing classical BPS backgrounds in SYM, namely solutions to the equations of motion that saturate the BPS bound and maintain a fraction of supersymmetry. The BPS properties are required again because the AdS/CFT duality is a strong-weak coupling duality; to check it one needs to be able to interpolate between the weak and strong couplings, and a fraction of unbroken supersymmetry might allow one to do so. The classical BPS backgrounds with given conserved charges usually have a moduli space for their collective coordinates. We suggest to do quantization on the moduli space of collective coordinates and expect that at least part of the resulting quantum states are BPS protected candidate states that we are looking for.

We restrict ourselves in this paper to scalar backgrounds, which can be viewed as Bose-Einstein condensates (BEC). It has been an old folklore in theoretical physics community that classical geometry, in a certain sense, can be viewed as a sort of BEC. So we think it natural to relate the geometric backgrounds in gravity to the scalar backgrounds in the

SYM dual. For $\mathcal{N} = 4$ SYM on $R \times S^3$, the classical scalar vacuum is unique because of the conformal coupling to the curvature of S^3, so the only available BEC-like objects are space-filling Q-balls [8] with conserved R-charge; we call them simply R-balls. The above considerations led us to examine the R-ball solutions that preserve 1/2, 1/4 or even less supersymmetry. To formulate the R-ball approach in this paper, we aim at 1) formulating the general conditions for BPS backgrounds in SYM; 2) finding explicitly classical BPS R-balls, particularly new solutions corresponding to non-commutative geometry in the large N limit; 3) carrying out collective-coordinate quantization on the moduli space of certain BPS R-balls; 4) showing the emergence of the known IQH droplet of fermions in a certain sector of half BPS R-ball configurations; and 5) finally showing that the new non-commutative BPS R-balls, upon quantization, lead to fractional quantum Hall (FQH) states, thus lending support to our previous argument [9] for the possible appearances of FQH-like states on the gravity side.

The success in carrying out the above steps 1)–3) will lay down the foundation for a general framework for finding classical BPS backgrounds in $\mathcal{N} = 4$ SYM on $R \times S^3$, and for constructing the quantum BPS states living on their moduli space (with given conserved charges), particularly in the large N limit. It is expected that the distinctive features due to the fact that the conformal field theory is defined on a compact space S^3 will play a decisive role in these discussions. The above points 4) and 5) will dwell on the relevance of the IQH and FQH states in the context of the holographic gravity/gauge duality. It has been noticed in the literature that the many-fermion state in phase space looks like an IQH droplet. Despite this, there is a crucial difference between the two, since the IQH droplet pre-requires the presence of the Landau levels and the projection to the lowest Landau level, while the former does not. If indeed it makes sense to talk about "Landau levels", then new states other than the IQH droplet can emerge due to the inclusion of interactions, giving rise to new candidates for non-perturbative states on the gravity side. The authors of the present paper have put forward arguments from the gravity side supporting the emergence of FQH-like states [9]. The essence of the arguments was the following: The interactions between the giant graviton probes in the LLM geometry background are shown to be repulsive; if the interactions can be extrapolated to finite density, then the giant gravitons in the LLM geometry at right densities can condense into new incompressible QH fluids with fractional filling factors. More concretely, the dynamics of giant graviton probes is first shown [9] to be described by a non-commutative Chern-Simons gauge theory [10]. Then it was further reduced to a non-commutative Chern-Simons matrix mechanics (NCCSMM) previously proposed in ref. [11], and its spectrum was shown to contain not only the IQH but also the FQH states. This has inspired us to try to find the FQH-like states on the gauge theory side. But this did not seem easy in the matrix model approach [2, 12]. (For other effort in studying the QHE in SYM, see refs. [13, 14].) Actually this was our main motivation to look for a new approach to the BPS dynamics in $\mathcal{N} = 4$ SYM. Indeed as shown below, new non-commutative BPS backgrounds can be found in our new framework in the large N limit, confirming the relevance of the FQHE and, more generally, of the non-commutative geometry to emergent gravity in SYM.

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This paper is organized as follows. We start with a brief review of supersymmetry transformations in $\mathcal{N} = 4$ SYM on $R \times S^3$ in section 2, to set up the notations and to formulate the conditions for unbroken supersymmetry. In section 3, we present some variational theorems for the classical BPS R-balls, i.e. classical scalar configurations of the lowest energy in the sector with given R-charges and leaving part of the supersymmetry unbroken. Then in section 4, we proceed to construct the explicit solutions for classical R-balls, which include the commutative half BPS configurations known in the literature. In particular, we find that in the large N limit there may exist non-commutative solutions, which solve the Gauss's constraints exactly and satisfy the BPS bound with an error of order $O(1/N)$. Thus the moduli space of the BPS R-balls is enhanced in the large N limit. We demonstrate this by presenting a new family of quarter BPS R-balls that involve a non-commutativity parameter between scalar and pseudo-scalar in pairs. In section 5, we discuss first in great detail the collective coordinate (or moduli space) quantization of the commutative half BPS R-balls. In particular, besides demonstrating how our new approach reproduces the known results in the half BSP sector, we show that the quantization naturally leads to the "Landau-level problem", so that it makes sense to look for more exotic FQH (or FQH-like) states in more complicated sectors. Indeed in section 6 we are able to show that the quantization of the non-commutative quarter BPS R-balls leads to an NCCSMM model for infinite-dimensional matrices, whose Hilbert space indeed contains FQH-like states. Section 7 is devoted to summary and discussions. Finally, in appendix of this paper, we derive the $\mathcal{N} = 4$ supersymmetry algebra on $R\times S^3$ and present the formula for the BPS bound, which assures that for our R-balls, the BPS bound is saturated by their R-charge, exactly corresponding to the BPS bound in the gravity dual.

2. $\mathcal{N} = 4$ SYM on $R \times S^3$

It is known that the $\mathcal{N} = 4$ supersymmetric vector multiplet in four dimensions can couple to a background metric in a Weyl invariant manner classically [15]. However, in quantum theory there exists Weyl anomaly except for certain symmetric backgrounds, such as $R \times S^3$ [16]. Accordingly, the $\mathcal{N} = 4$ SYM on $R \times S^3$ is a well-defined quantum conformal field theory; the corresponding Lagrangian in $\mathcal{N} = 1$ language reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{Tr}(D_\mu X_i D^\mu X^i + D_\mu Y_i D^\mu Y^i) - V(X,Y)\\ &+\frac{i}{2}\mathrm{Tr}(\bar\psi\gamma^a e^\mu_a D_\mu\psi) - \frac{ig}{2}\mathrm{Tr}\{\bar\psi(\alpha^i[X_i,\psi] + \gamma^5\beta^j[Y_j,\psi])\}.\end{aligned} \tag{2.1}$$

Here g is the gauge coupling; the potential of scalars is given by

$$V(X,Y) = \frac{1}{2R^2}\mathrm{Tr}(X_iX^i + Y_jY^j) - \frac{g^2}{4}\mathrm{Tr}([X_i,X_j]^2 + [Y_i,Y_j]^2 + 2[X_i,Y_j]^2) \tag{2.2}$$

with R the radius of S^3; X and Y denote scalar and pseudo-scalar fields respectively. α^i, β^j $(i,j = 1,2,3)$ are 4×4 real anti-symmetric matrices satisfying the algebraic relations [17]

$$\begin{aligned}&[\alpha^i,\alpha^j] = -2\epsilon^{ijk}\alpha^k, \qquad [\beta^i,\beta^j] = -2\epsilon^{ijk}\beta^k,\\ &\{\alpha^i,\alpha^j\} = \{\beta^i,\beta^j\} = -2\delta^{ij}, \qquad [\alpha^i,\beta^j] = 0.\end{aligned} \tag{2.3}$$

All gamma matrices are defined in a local Lorentz frame e_a^μ and D_μ is the covariant derivative:

$$D_\mu\psi = \nabla_\mu\psi - ig[A_\mu,\psi] = \partial_\mu\psi + \Omega_\mu\psi - ig[A_\mu,\psi], \tag{2.4}$$

where $\Omega_\mu = \frac{1}{4}\Omega_\mu^{ab}\gamma_{ab}$ is the spin connection with Ω^{ab} the connection 1-form with γ_{ab} the generators of local Lorentz transformations. Two remarks are deserved here. First, it is the cylinder $R\times S^3$, instead of Minkowski space M^4, that is the global conformal boundary of AdS_5. Second, because of the cylindrical structure of $R\times S^3$, the natural spin connection does not involve the temporal direction, implying $\nabla_0 = \partial_0$; as a result, only local SO(3) transformations, instead of local SO(3, 1) transformations, act upon the gaugino field ψ; namely, there are no local boosts. For this reason, a global and Majorana formalism for ψ and γ_a can be defined on $R\times S^3$ similar to that in flat space. More subtleties of the spin structure on $R\times S^3$ will be explored in the following discussion of global supersymmetry.

The action integral of the Lagrangian (2.1) over $R\times S^3$ possesses an $\mathcal{N}=4$ superconformal symmetry because of the existence of four conformal Killing spinors ϵ_A for $A=1,2,3,4$. We will first follow the analysis in refs. [19, 18], which dealt with the conformal Killing spinors on $R\times S^3$ by descending the Killing spinors on AdS_5. In this formalism, the conformal Killing spinor equations are written as

$$\partial_0\epsilon_A = \frac{i}{2R}\Gamma_0\epsilon_A, \qquad \nabla_m\epsilon_A = \frac{i}{2R}\Gamma_m\Gamma_5\epsilon_A\,, \tag{2.5}$$

where $m=1,2,3$ labels the directions in a local orthonormal frame on S^3, and the five upper-cased gamma matrices generate the Clifford algebra of five-dimensional Minkowski space M^5, which form a local frame on AdS_5. To the best, there are four linear-independent complex global sections in the spin bundle on $R\times S^3$ as the solutions to eq. (2.5) for each $A=1,2,3,4$; therefore, there are at most 32 supercharges.

In addition to the above *extrinsic* formalism (descending from AdS_5 in eq. (2.5)), there is an *intrinsic* formalism for the spin structure on $R\times S^3$, by making use of Majorana spinors, in which the gamma matrices are denoted with the lower case. In fact, the two formalisms share the following feature: There is only one local SO(3) acting on the four-component complex spinor ϵ_A. This implies that the four-component spinor ϵ_A in eq. (2.5) as a representation of this local symmetry must be reducible. A natural reduction results from the observation that $[\Gamma_0,\Gamma_m\Gamma_5]=0$. One can introduce a projection operator $\mathcal{P} = (1+i\Gamma_0)/2$ such that a "Γ_0-chirality" is defined as follows: (for convenience, the subscript A is omitted for a moment)

$$\epsilon_L := \mathcal{P}\epsilon, \quad \epsilon_R := (1-\mathcal{P})\epsilon. \tag{2.6}$$

Two Majorana spinors in the intrinsic spin structure can be constructed from ϵ_L or ϵ_R, respectively, via the standard procedure that produces a Majorana spinor from a Weyl spinor:

$$\zeta_L = \epsilon_L + \mathcal{C}\epsilon_L^*, \quad \zeta_R = \epsilon_L + \mathcal{C}\epsilon_R^*, \tag{2.7}$$

where $\mathcal{C}$ is the conventional charge conjugation matrix. Note that the conformal Killing spinor ϵ in the (extrinsic) AdS-descending spin structure does not admit any Majorana condition.

With the above-mentioned clarification of the spin structure on $R \times S^3$, we can write down the fermionic part of $\mathcal{N} = 4$ superconformal transformation explicitly [18, 16]:

$$\begin{aligned}
\delta_{L,R} A_\mu &= -i\bar{\psi}\gamma_\mu \zeta_{L,R}, \qquad \delta_{L,R} X_i = \bar{\psi}\alpha^i \zeta_{L,R}, \qquad \delta_{L,R} Y_j = i\bar{\psi}\beta^j \gamma_5 \zeta_{L,R}, \\
\delta_{L,R}\psi &= \frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}\zeta_{L,R} - i\gamma^\mu(\alpha^i D_\mu X_i + i\gamma_5 \beta^j D_\mu Y_j)\zeta_{L,R} \\
&\quad + \frac{ig}{2}\epsilon^{ijk}(\alpha^k [X_i, X_j] + \beta^k [Y_i, Y_j])\zeta_{L,R} + g[X_i, Y_j]\alpha^i \beta^j \gamma_5 \zeta_{L,R}, \\
&\quad - \frac{i}{2}(\alpha^i X_i - i\gamma_5 \beta^j Y_j)\gamma^\mu \nabla_\mu \zeta_{L,R}\,,
\end{aligned} \tag{2.8}$$

where, as usual, $\gamma^\mu = \gamma^a e_a^\mu$ with e_a^μ the local vierbein. As R is sent to infinity, both the Lagrangian (2.1) and the superconformal transformation (2.8) reduce to those in M^4. Using the conformal Killing spinor equations (2.5), we have

$$\gamma^\mu \nabla_\mu \zeta_L = \frac{2i}{R}\gamma_5 \gamma^0 \zeta_L, \qquad \gamma^\mu \nabla_\mu \zeta_R = -\frac{2i}{R}\gamma_5 \gamma^0 \zeta_R. \tag{2.9}$$

Then one can directly check that the variation of the Lagrangian (2.1) is indeed a total derivative. We will show in appendix that either ζ_L or ζ_R generates a super-isometry algebra separately; together, they generate the entire superconformal algebra.

3. R-balls as classical BPS backgrounds

As mentioned in the introduction, recent progress in understanding emergent gravity in AdS/SYM holography motivated us to look for scalar field configurations (representing a sort of BEC) that preserve a fraction of supersymmetry. In this attempt, the old idea of Coleman's Q-balls [8] has attracted our attention. Initially, Q-balls are defined as non-topological soliton solutions in complex scalar field theories in a four-dimensional flat space-time. Their existence and classical stability hinge on the existence of a conserved charge, Q, associated with a global U(1) symmetry: A Q-ball is the solution minimizing the energy in the sector with a fixed and sufficiently large Q-charge. The solutions constructed in ref. [8] are spherically symmetric in space, and the nonzero Q-charge is generated by rotating a static configuration in internal space. In our present case, we generalize Q-ball to the compact space S^3; this generalization allows the existence of *space-filling* Q-balls, which is impossible in non-compact flat space. In $\mathcal{N} = 4$ SYM, there are two important complications for Q-balls. First, the pertinent global symmetry is SO(6) R-symmetry. The corresponding Q-balls, which we call *R-balls*, carry a U(1) R-charge embedded in the non-abelian SO(6). Different embedding yields a different type of R-balls. Second, the theory has a color U(N) gauge symmetry, and the scalar fields also carry color degrees of freedom. The Gauss's law will severely constrain possible physical states after quantization. In this section, we will formulate and analyze the conditions for the classical R-balls, namely solutions to the equations of motion with energy saturated by the R-charge in a given sector with fixed R-charge.

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3.1 Variational BPS bound

First we consider the variational aspects of the problem. As usual for classical backgrounds, we set the gaugino field to zero: $\psi = 0$. Then the Hamiltonian reads:

$$H = \int_{S^3} \left(\frac{1}{2} \mathrm{Tr} \left(F_{0i}^2 + \frac{1}{2} F_{ij}^2 + D_i \phi_s D_i \phi_s + D_0 \phi_s D_0 \phi_s + \frac{1}{R^2} \phi_s \phi_s \right) + V_4(\phi) \right), \tag{3.1}$$

where $\{\phi_s\} = \{X_a, Y_a\}$ (with $s = 1, \ldots, 6$ and $a = 1, 2, 3$) are six scalar fields, transforming as a vector under R-symmetry SO(6), while as adjoint representation under color U(N). We note that all terms in eq. (3.1) are non-negative. To look for the Q-ball solutions, we concentrate on the dynamics of the scalar fields. Thus we set $F_{\mu\nu} = 0, (\mu, \nu = 0, 1, 2, 3)$, putting the first two terms to zero. Because S^3 is simply connected, it is possible to take the spatial components of the gauge potential $A_i = 0$, while allowing A_0 a function of time only. In the following, we will adapt the BPS analysis to the global R-symmetry.

Let us focus on a sector of scalar configurations with a fixed U(1) R-charge:

$$Q_{\mathbf{r}} = \mathrm{Tr} \int_{S^3} (D_0 \phi_s) \mathbf{r}_{st} \phi_t, \tag{3.2}$$

associated with a generator $\mathbf{r}$ in the $so(6)$ Lie algebra (in the definition representation). Note that as a six-by-six *antisymmetric* matrix, $\mathbf{r} = (\mathbf{r}_{st})$ may be degenerate. If $\mathbf{r}$ is also an *orthogonal* matrix in a linear subspace in which it is non-degenerate, then it is easy to prove the following BPS-like inequality:

$$H \geq \frac{1}{R} \left| \mathrm{Tr} \int_{S^3} (D_0 \phi_s) \mathbf{r}_{st} \phi_t \right| = \frac{|Q_{\mathbf{r}}|}{R}. \tag{3.3}$$

The energy H saturates the lower bound set by the charge $Q_{\mathbf{r}}$ in (3.3) only when the following three conditions are satisfied: First,

$$D_i \phi_s = 0, \quad (i = 1, 2, 3), \tag{3.4}$$

which makes the third term in eq. (3.1) vanish; and

$$[\phi_s, \phi_t] = 0, \tag{3.5}$$

making the contribution of the quartic potential V_4 vanish; and finally

$$D_0 \phi_s = \pm R^{-1} \mathbf{r}_{st} \phi_t, \tag{3.6}$$

with ϕ_s having *only non-zero* components in the subspace in which the generator $\mathbf{r}$ is non-degenerate. Since $A_i = 0$, (3.4) implies that the scalar fields ϕ_s are constant, i.e. the lowest KK modes, on S^3. The condition (3.6) means the time-dependent configuration $\phi_s(t)$ rotates in internal space with a specific frequency, which generates the R-charge that saturates the lower energy bound. Note that after imposing $F_{\mu\nu} = 0$ and eq. (3.4), the Hamiltonian is reduced to a gauged mechanics:

$$H = \int_{S^3} \left\{ \mathrm{Tr} \left(D_0 \phi_s D_0 \phi_s + \frac{1}{R^2} \phi_s \phi_s \right) + V_4(\phi) \right\}. \tag{3.7}$$

The conditions (3.4) and (3.5), we will call the BPS conditions, are those to saturate the energy bound in a given sector with definite R-charge.

In the next section, we will see that the configurations satisfying these BPS conditions automatically preserve part of the $\mathcal{N}=4$ supersymmetry. Moreover, we note that when the commutative condition (3.5) is satisfied, the Gauss's law constraint is also automatically satisfied, since the color charge density vanishes:

$$j^0_{\mathrm{U}(N)} = [D_0\phi_s, \phi_s] = \pm R^{-1}\mathbf{r}_{st}[\phi_t, \phi_s] = 0. \tag{3.8}$$

We will call the Q-ball solutions to the equations of motion obtained by solving the BPS equations (3.6) and (3.5) as BPS R-balls. As zero-modes on S^3, they are space-filling. They form a decoupled sector in the limit when the radius R of S^3 tends to zero. (Note that non-space-filling R-ball configurations exist on S^3, but they may not be BPS in the sense that the energy is not saturated by their R-charge. This is in accordance with ref. [20]. In the same limit, the gaugino and gluon backgrounds are decoupled from this sector too [16].) Here we would like to warn that the BPS eq. (3.6) does not have a topological origin, since the R-charge is not central in SUSY algebra. Incidentally, we also make the remark that topological BPS solitons, such as 't Hooft-Polyakov monopoles and dyons with non-vanishing charges, do not exist on the compact S^3. This is the main reason why we turn our attention to non-topological R-balls in search of BEC-like backgrounds.

3.2 Group theory considerations

By an SO(6) rotation the antisymmetric $\mathbf{r}$-matrix can always be put in the following canonical form:

$$\mathbf{r} \to \mathbf{r}_{\mathrm{can}} = \begin{pmatrix} & -r_1 & & & & \\ r_1 & & & & & \\ & & & -r_2 & & \\ & & r_2 & & & \\ & & & & & -r_3 \\ & & & & r_3 & \end{pmatrix}. \tag{3.9}$$

where, since the $\mathbf{r}$-matrix has to be an orthogonal matrix in a subspace in which it is non-degenerate, there are only four choices for r_α:

$$(r_1, r_2, r_3) \in \{(1,0,0), (1,1,0), (1,1,1), (1,1,-1)\}. \tag{3.10}$$

Any different choice of "gauge" is equivalent to them. For example, the second solution can be chosen as well to be $(1,-1,0)$. However, since SO(6) can not be enlarged to be $O(6)$ as the global symmetry for the $\mathcal{N}=4$ SYM, the last two solutions are *not* equivalent. In the next section we will see that, if the number of the non-vanishing r_α's is one, two or three, respectively, the corresponding R-ball states maintain 1/2, 1/4 or 1/8 supersummetry.

3.3 Non-commutative solutions at large N

In color space, the scalar fields ϕ_s are N-by-N matrices. The commutative ansatz (3.5) has been used before in ref. [2] to define a holomorphic normal matrix model. The above approach allows us to consider more sophisticated R-balls by going beyond this ansatz but still having good control. The simplest case is that in the large N limit, the commutators $[\phi_s, \phi_t]$ are proportional to the unit matrix in color space:

$$[\phi_s, \phi_t] = i\frac{\theta_{st}}{R^4}, \tag{3.11}$$

where the non-commutative (NC) parameters θ_{st} are anti-symmetric and of the dimension of length squared. Other ansatz or conditions in the last subsections are unchanged, except the Gauss's law.

By taking derivative, one has

$$[D_0\phi_s, \phi_t] + [\phi_s, D_0\phi_t] = i\frac{\dot{\theta}_{st}}{R^4}. \tag{3.12}$$

By eq. (3.6) and the antisymmetry of $\mathbf{r}$, we get the equation of motion for the NC parameter:

$$\dot{\Theta} = \pm[\mathbf{r}, \Theta]\,, \tag{3.13}$$

where Θ is the matrix (θ_{st}). For simplicity, in this paper we will only consider the case with $\dot{\Theta} = 0$, or equivalently $[\mathbf{r}, \Theta] = 0$. Accordingly, in the same basis for (3.9), Θ can be put in a canonical form:

$$\Theta \to \Theta_{\rm can} = \begin{pmatrix} & -\theta_1 & & & & \\ \theta_1 & & & & & \\ & & & -\theta_2 & & \\ & & \theta_2 & & & \\ & & & & & -\theta_3 \\ & & & & \theta_3 & \end{pmatrix}. \tag{3.14}$$

In addition, one must keep in mind that, for any $r_\alpha = 0$, the corresponding $\theta_\alpha = 0$. Moreover, the Gauss's law requires

$$[D_0\phi_s, \phi_s] = \pm\frac{i}{R^5}\mathrm{Tr}_{6\times 6}(\mathbf{r}\Theta) = 0; \tag{3.15}$$

in the canonical forms for $\mathbf{r}$ and Θ, we have

$$\sum_{\alpha=1}^{3}\theta_\alpha r_\alpha = 0. \tag{3.16}$$

There are two straightforward implications: (i) If there is only one $r_\alpha \neq 0$, then the deformation (3.11) with constant θ violates the Gauss's law. (ii) If two $r_\alpha \neq 0$ and we choose the gauge $r_1 = -r_2 = 1$, then $\theta_1 = \theta_2$ and $\theta_3 = 0$. In the second half of the next section, we will look for R-balls in the case (ii).

The deformation (3.11) is self-consistent only if $N \to \infty$. Then various sums over colors in the Hamiltonian become divergent. To define a well-behaved large N limit, one needs to properly redefine the the trace, Tr, and the fields by rescaling them with some negative powers of N. In fact, let us examine the relevant terms in H:

$$H = 2\pi^2 R^3 \mathrm{Tr}\left(\frac{\phi_s^2}{R^2} - \frac{g^2}{4}[\phi_s, \phi_t]^2\right), \tag{3.17}$$

where the factor $2\pi^2 R^3$ is the volume of S^3 with radius R. Now we perform the standard large N trick, by redefining $\phi_s = \sqrt{N}\tilde{\phi}_s$, $\lambda = g^2 N$. Then we extract the scale from Θ, write it as θ and identify the dilatation operator with

$$\Delta = \frac{R}{2\pi^2}H = N\mathrm{Tr}\left(\frac{\theta}{R^2}\varphi_s^2 - \frac{\lambda}{4}\left(\frac{\theta}{R^2}\right)^2[\varphi_s, \varphi_t]^2\right), \tag{3.18}$$

where $\varphi_s = R^2\tilde{\phi}_s/\sqrt{\theta}$ and the pre-factor N plays the role of $\hbar^{-1}$. The concrete color space "renormalization" scheme will be specified for different choices of Θ. We will consider two examples below.

First, if $\theta = |\theta_1| \neq 0$ and $\theta_{2,3} = 0$, then we set the following order estimation: $\mathrm{Tr}\varphi_s^2 \sim \mathcal{O}(N^2)$ while $[\varphi_s, \varphi_t] \sim \mathcal{O}(1)$, which is in consistence with (3.11). Then it is natural to take $N^2\theta/R^2$ fixed, written as c_1; the dilatation operator becomes

$$\Delta = N\left(c_1\frac{\mathrm{Tr}}{N^2}(\varphi_s^2) - \frac{\lambda c_1^2}{4}\frac{\mathrm{Tr}}{N^4}([\varphi_s, \varphi_t]^2)\right). \tag{3.19}$$

Notably, the symbols $\mathrm{Tr/N^2}$ play the role of the regularized traces. Moreover, the ratio of the quartic to the quadratic terms in eq. (3.19) is of order $\lambda c_1 N^{-3}$. We define the 't Hooft coupling λ and c_1 to be the physical parameters independent of the cutoff N, then the quartic terms becomes irrelevant. Meanwhile, the BPS bound is again saturated.

For the second example, let us consider the case with $\theta_1 = \theta_2$ with absolute value θ and $\theta_3 = 0$. The solution to eq. (3.11) has a direct product structure (see eqs. (4.11) and (4.17) in next section), and two regulators N_1 and N_2 are needed such that $N_1N_2 = N$. Then the order estimation is set to be $\mathrm{Tr}\varphi_s^2 \sim \mathcal{O}(N(N_1 + N_2))$. In this case, the physical parameter is given by $c_2 = N(N_1 + N_2)\theta/R^2$ and the physical dilatation is

$$\Delta = N\left(c_2\frac{\mathrm{Tr}(\varphi_s^2)}{N(N_1 + N_2)} - \frac{\lambda c_2^2}{4}\frac{\mathrm{Tr}([\varphi_s, \varphi_t]^2)}{N^2(N_1 + N_2)^2}\right). \tag{3.20}$$

This time, $\mathrm{Tr/N(N_1 + N_2)}$ serve as the regularized trace; the ratio of the quartic to the quadratic terms in eq. (3.20) is of order $\lambda c_2/N(N_1 + N_2)^2$ with c_2 and λ to be fixed, independent of N. Note that in both (3.19) and (3.20), one should *not* further absorb c_1 and c_2 into another redefinition of fields, because this would change the universality classes of the model.

In summary, with the non-commutative ansatz (3.11), the BPS bound (3.3) is saturated up to an error that vanishes in a large N limit. We will refer this type of configurations as *almost-BPS*. Allowing us to see this new possibility is a significant advantage of our approach.

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4. R-ball solutions with unbroken SUSY

With only bosonic backgrounds, unbroken supersymmetry requires $\delta_\zeta \psi = 0$. In this section we will find the BPS R-ball configurations that also preserve part of supersymmetry, i.e. satisfy the conditions (from (2.8))

$$\begin{aligned}
0 = \delta_L \psi_a &= -i\gamma^0(\alpha^i \dot{X}_{ia} + i\gamma_5 \beta^j \dot{Y}_{ja})\zeta_L + \frac{1}{R}(\alpha^i X_{ia} - i\gamma_5\beta^j Y_{ja})\gamma_5\gamma^0\zeta_L \\
&\quad + \frac{ig}{2}\epsilon^{ijk}(\alpha^k [X_i, X_j]_a + \beta^k [Y_i, Y_j]_a)\zeta_L + g[X_i, Y_j]_a \alpha^i\beta^j\gamma_5\zeta_L, \\
0 = \delta_R \psi_a &= -i\gamma^0(\alpha^i \dot{X}_{ia} + i\gamma_5 \beta^j \dot{Y}_{ja})\zeta_R - \frac{1}{R}(\alpha^i X_{ia} - i\gamma_5\beta^j Y_{ja})\gamma_5\gamma^0\zeta_R \\
&\quad + \frac{ig}{2}\epsilon^{ijk}(\alpha^k [X_i, X_j]_a + \beta^k [Y_i, Y_j]_a)\zeta_R + g[X_i, Y_j]_a \alpha^i\beta^j\gamma_5\zeta_R,
\end{aligned} \tag{4.1}$$

with $\zeta_{L,R}$ not all vanishing. Here $a = 0, 1, 2, \ldots, N^2 - 1$ are the indices of the adjoint U(N) representation. A representation of the algebra (2.3) is chosen to be

$$\begin{aligned}
&\alpha_1 = i\sigma_2 \times \sigma_1, \qquad \alpha_2 = -i\sigma_2 \times \sigma_3, \qquad \alpha_3 = i\mathbf{1}_{2\times 2} \times \sigma_2, \\
&\beta_1 = -i\sigma_1 \times \sigma_2, \qquad \beta_2 = -i\sigma_2 \times \mathbf{1}_{2\times 2}, \qquad \beta_3 = i\sigma_3 \times \sigma_2.
\end{aligned} \tag{4.2}$$

In this section, we will not only find classical solutions satisfying these conditions, but also count the moduli of the solutions of a given type. A clear understanding of the moduli of the solution space is crucial for the collective coordinate quantization we are going to apply in the next section. This is because the moduli form the configuration space of the collective coordinates for a given type of solutions, and the collective coordinate quantization heavily exploits the knowledge of the moduli space. To avoid overcounting, one needs to be careful: SO(6) inequivalent configurations may be gauge equivalent, since the scalars carry both global SO(6) and local U(N) degrees of freedom, which may be entangled in the moduli counting.

4.1 Commutative R-balls and their moduli

We first consider the commutative ansatz (3.5) with only $r_1 \neq 0$ and $r_2 = r_3 = 0$ in the canonical form (3.9). In this case, we need to consider only one pair of scalar fields $X = X_1$ and $Y = Y_1$. Then the supersymmetry condition (4.1) reduces to two systems of linear equations:

$$\begin{cases}
G_a\zeta_1 + (K_a - F_a)\zeta_4 = 0, \\
G_a\zeta_2 - (K_a + F_a)\zeta_3 = 0, \\
(K_a + F_a)\zeta_2 + G_a\zeta_3 = 0, \\
(K_a - F_a)\zeta_1 - G_a\zeta_4 = 0,
\end{cases} \quad \text{and} \quad
\begin{cases}
G_a\tilde{\zeta}_1 + (\tilde{K}_a - \tilde{F}_a)\tilde{\zeta}_4 = 0, \\
G_a\tilde{\zeta}_2 - (\tilde{K}_a + \tilde{F}_a)\tilde{\zeta}_3 = 0, \\
(\tilde{K}_a + \tilde{F}_a)\tilde{\zeta}_2 + G_a\tilde{\zeta}_3 = 0, \\
(\tilde{K}_a - \tilde{F}_a)\tilde{\zeta}_1 - G_a\tilde{\zeta}_4 = 0,
\end{cases} \tag{4.3}$$

where ζ_A, $\tilde{\zeta}_A$ ($A = 1, 2, 3, 4$) are the "left" and "right" Majorana conformal Killing spinors and

$$\begin{aligned}
G_a = g\gamma_5[X, Y]_a, \qquad K_a &= -i\gamma^0\dot{X}_a + \gamma_5\gamma^0 X_a/R, \qquad F_a = \gamma^0\gamma_5\dot{Y}_a - i\gamma^0 Y_a/R, \\
\tilde{K}_a &= -i\gamma^0\dot{X}_a - \gamma_5\gamma^0 X_a/R, \qquad \tilde{F}_a = \gamma^0\gamma_5\dot{Y}_a + i\gamma^0 Y_a/R.
\end{aligned} \tag{4.4}$$

The vanishing determinant of each system leads to the conditions for unbroken supersymmetry:

$$\left(|-i\dot{Z}_a+R^{-1}Z_a|^2-\left(\frac{g}{2}[Z,Z^\dagger]_a\right)^2\right)\left(|i\dot{Z}_a+R^{-1}Z_a|^2-\left(\frac{g}{2}[Z,Z^\dagger]_a\right)^2\right)=0, \tag{4.5}$$

with $Z = X + iY$. It is easy to see that when $[X, Y] = 0$, i.e. $[Z, Z^\dagger] = 0$, the above equations are reduced to the BPS condition (3.6) in the $A_0 = 0$ gauge. The latter is easily solved, resulting in

$$Z = e^{\pm it/R} A\,, \tag{4.6}$$

with A any $N \times N$ time-independent normal matrices: $[A, A^\dagger] = 0$. Inserting the solutions (4.6) back into eq. (4.3), it is easy to verify that there are 16 supercharges. Hence the solutions (4.6) are 1/2-BPS backgrounds.

Other examples can be worked too. It can be verified that if the number of non-zero r_α's in the canonical form (3.9) is γ, then the fraction of unbroken supersymmetry is $1/2^\gamma$. An important question is how many commutative BPS R-balls there are. This is the problem of counting the moduli of such solutions, which we will address here.

With the canonical form (3.9) of $\mathbf{r}$, one may define complex scalars as $Z_\alpha = \phi_{2\alpha-1}+i\phi_{2\alpha}$ ($\alpha = 1, 2, 3$). Then the equations (3.4), (3.6) and (3.5) that completely determine the commutative BPS R-balls can be recast into the form:

$$\dot{Z}_\alpha = i\frac{r_\alpha}{R}Z_\alpha, \qquad [Z_\alpha, Z_\beta] = [Z_\alpha, Z_\beta^\dagger] = 0. \tag{4.7}$$

The first equation can be solved by $Z_\alpha = A_\alpha e^{ir_\alpha t/R}$ with A_α time-independent N-by-N matrices. The second equation indicates that A_α can be put in the form

$$A_\alpha = U^\dagger a_\alpha U \tag{4.8}$$

with a_α (the eigenvalue matrix) diagonal: $a_\alpha = diag(a_{\alpha,1}, a_{\alpha,2}, \ldots, a_{\alpha,N})$, and U unitary. Since all diagonal U's give rise to the same A_α when a_α is diagonal, the solution space for a particular R-charge generator $\mathbf{r}$ is given by

$$\mathcal{M}_{\mathbf{r},N} = \{(a_\alpha, U)\} = \mathbf{C}^{\gamma N} \times \mathrm{U}(N)/\,\mathrm{U}(1)^N\ . \tag{4.9}$$

(recall that γ is the number of non-zero r_α.) Furthermore, recall the BPS bound (3.3)

$$Q_{\mathbf{r}} = \pm\frac{1}{2\pi^2 R}\int_{S^3}\sum_\alpha \mathrm{Tr}(Z_\alpha^\dagger Z_\alpha) = \pm R^2\sum_{\alpha,i}|a_{\alpha,i}|^2\,, \tag{4.10}$$

where we have absorbed a volume factor $2\pi^2$ into the R-charge. Thus, an R-charge sector is a class of R-ball solutions with the same R-charge $Q_{\mathbf{r}}$ associated with generator $\mathbf{r}$. Once the R-charge generator is specified, the moduli space $\mathcal{M}_{\mathbf{r},N}$ is divided into different R-charge sectors, and in each sector the value of the dilatation Δ defined by eq. (3.18) is fixed by the BPS condition $\Delta = Q_{\mathbf{r}}$.

As an example, for the 1/2 BPS R-charge sector with $r_1 = 1$, $r_2 = r_3 = 0$, the R-charge Q is given by $Q = R^2\sum_i |a_i|^2$. Similarly, each commutative $1/2^\gamma$ BPS R-charge sector with a canonical $\mathbf{r}$ defines a sphere $S^{2\gamma N-1}$ in the eigenvalue space $\mathbf{R}^{2\gamma N}$.

4.2 Non-commutative R-balls and their moduli

Now we consider the non-commutative ansatz (3.11), which leads to new solutions in large N. We have shown that for a single non-vanishing complex scalar Z, the non-commutative, almost-BPS background violates the Gauss's law (3.15), since $[\dot{Z}, Z^\dagger] + [\dot{Z}^\dagger, Z] \neq 0$. So we need, at least, to turn on two complex scalars: $Z_1 = X_1 + iY_1$, $Z_2 = X_2 + iY_2$. Based on the discussions in the previous section, we should pick $r_1 = 1$ and $r_2 = -1$ for the R-charge generator. With this choice we have the solutions

$$Z_1 = \frac{1}{R^2} e^{it/R} A_1, \quad Z_2 = \frac{1}{R^2} e^{-it/R} A_2, \tag{4.11}$$

where the time-independent matrices A_1 and A_2 obey a two-dimensional Heisenberg algebra:

$$\begin{aligned} &[A_1, A_1^\dagger] = 2\theta_1, \qquad\qquad\qquad [A_2, A_2^\dagger] = 2\theta_2, \\ &[A_1, A_2] = [A_1, A_2^\dagger] = [A_2, A_1^\dagger] = [A_1^\dagger, A_2^\dagger] = 0, \end{aligned} \tag{4.12}$$

with $\theta_1 = \theta_2$, $|\theta_1| = \theta$. The matrices A_1 and A_2 span two orthogonal non-commutative planes and the Gauss's law constraint is satisfied because $\theta_1 = \theta_2$.

Now let us count the number of unbroken supersymmetries via solving the supersymmetry variation condition (4.1). In this case, this condition reduces to:

$$\begin{cases} (G_{11} - G_{22})_a \zeta_1 - (K_2 + F_2)_a \zeta_3 + (K_1 - F_1)_a \zeta_4 = 0, \\ -(G_{11} - G_{22})_a \zeta_2 + (K_1 + F_1)_a \zeta_3 + (K_2 - F_2)_a \zeta_4 = 0, \\ (K_2 + F_2)_a \zeta_1 - (K_1 + F_1)_a \zeta_2 - (G_{11} + G_{22})_a \zeta_3 = 0, \\ -(K_1 - F_1)_a \zeta_1 - (K_2 - F_2)_a \zeta_2 + (G_{11} + G_{22})_a \zeta_4 = 0, \end{cases} \tag{4.13}$$

in which

$$G_{ij} = g\gamma_5 [X_i, Y_j], \qquad K_i = -i\gamma^0 \dot{X}_i + \gamma_5 \gamma^0 X_i / R, \qquad F_i = \gamma^0 \gamma_5 \dot{Y}_i - i\gamma^0 Y_i / R \tag{4.14}$$

for $i, j = 1, 2$, with color indices suppressed. Since the solution (4.11) and the commutation relation (4.12) lead to

$$G_{11} - G_{22} = K_1 + F_1 = K_2 - F_2 = 0, \tag{4.15}$$

the solution of (4.13) is $\zeta_{1,3,4} = 0$, leaving ζ_2 the only surviving Killing spinor. Similar result holds for $\tilde{\zeta}_A$. So among all 32 components of Killing spinors, only a quarter of them can be nonzero and linearly independent. Therefore, the classical non-commutative configurations (4.11) with (4.12) preserve eight supersymmetries. This class of 1/4 BPS backgrounds has not been discovered before in the literature.

The moduli space, $\mathcal{M}_{\mathbf{r},\Theta}$, for this class of non-commutative R-balls is qualitatively different from that of the commutative BPS R-balls. To start, we rewrite the non-commutative ansatz (3.11) in the exponential form:

$$\exp(i\phi_s u^s) \exp(i\phi_t v^t) = e^{-\theta_{st} u^s v^t / R^4} \exp(i\phi_t v^t) \exp(i\phi_s u^s) \,, \tag{4.16}$$

where u^s and v^t are two vectors in R^6. By the celebrated Stone-von Neumann theorem, any solutions to (4.16) are unitarily equivalent. Then we focus on the case of (4.11). Because r_1 and r_2 is now gauged to be 1 and -1, the sign of $\theta_1 = \theta_2$ matters; so there are actually two different solutions (θ, θ) and $(-\theta, -\theta)$, where θ by definition is non-negative. Without losing of generality, we take (θ, θ). In this case,

$$A_\alpha = \sqrt{2\theta} U^\dagger a_\alpha U, \;\; a_1 = a \times \mathbf{1}, \;\; a_2 = \mathbf{1} \times a \,, \tag{4.17}$$

where a is a standard matrix representation in quantum mechanics

$$a = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & \ddots & \sqrt{3} & \\ & & & \ddots & \ddots \end{pmatrix} \tag{4.18}$$

and U is an infinite-dimensional unitary matrix. So $\mathcal{M}_{\mathbf{r},\Theta}$ is the product of $\mathbf{R}^+ = \{\sqrt{\theta}\}$ and an infinite special unitary group "SU(∞)", loosely speaking. However, the difference with the commutative half-BPS case is that only a U(2) subgroup in this SU(∞) will be considered as dynamical variables, with the rest being gauge degrees of freedom. This U(2) is generated by the Schwinger representation,

$$\mathbf{L}_\mu = a^\dagger_\alpha (\sigma_\mu)^\alpha_\beta a^\beta, \tag{4.19}$$

with $\mu = 0, 1, 2, 3$, $\sigma_0 = 1$. It is easy to show $\mathbf{L}_{1,2}$ do not contribute to the energy in (3.20). According to the above-mentioned analysis, we reparameterize this type of R-balls as

$$A_\alpha = \sqrt{2\theta} e^{i\varphi_\alpha} V^\dagger a_\alpha V, \qquad \alpha = 1, 2, \tag{4.20}$$

with $V \in \mathrm{SU}(\infty)/\,\mathrm{U}(2)$. $\mathbf{L}_0$ generates the translation in the direction of $\varphi_1 + \varphi_2$ while $\mathbf{L}_3$ generates the translation in the direction of $\varphi_1 - \varphi_2$. And in the later treatment, $\sqrt{\theta}$, φ_α are dynamical while V plays the role of gauge degrees of freedom.

Now let us regularize a_1 and a_2 by truncating (4.18) to estimate the N-dependence of the R-charge. The first factor in the direct product (4.17) is regularized by the upper-left N_1-by-N_1 block, while the second factor by the upper-left N_2-by-N_2 block. Accordingly, we have $N = N_1 N_2$. The R-charge calculated from eq. (3.20) (with the quartic term safely thrown away) is given by

$$Q = N c_2 = N^2 (N_1 + N_2) \frac{\theta}{R^2}. \tag{4.21}$$

The prefactor N is familiar in any quadratic quantities in large N field theories. Now the R-ball sector is specified by the "renormalized R-charge" c_2, instead of the bare ratio θ/R^2. In the later treatment, we will omit the prefactor N and concentrate on the finite part of the physical quantities like energy and charge.

5. Quantization of the commutative half BPS sector

The classical R-balls have continuous values for the U(1) R-charge. It is necessary to quantize the R-balls in order to get a discrete spectrum for the R-charge. In this section, we will quantize the R-balls by both collective coordinate quantization [22 – 24] and canonical quantization. In the commutative half BPS sector we will show that both quantization reproduce the previous results obtained by the matrix model approach [2], but our treatment will shed new light on several important aspects of physics. In particular, we can explicitly exhibit the origin of the Landau levels in the present text, so as to make the connection of the BPS dynamics with the quantum Hall effect meaningful and substantial.

5.1 Collective coordinate quantization

We have seen the time-independent matrix A in our solution (4.6) can be put in the form

$$A = U^\dagger a U, \tag{5.1}$$

where $a = diag(a_1, a_2, \ldots, a_N)$ and $U \in \mathrm{U}(N)/\,\mathrm{U}(1)^N$. Consequently, the collective coordinate space for solution (4.6) is identified to be

$$\mathcal{M}_{\mathbf{r},N} = \left(\mathbf{C}^N/S_N\right) \times \left(\mathrm{U}(N)/\,\mathrm{U}(1)^N\right). \tag{5.2}$$

Here S_N is the symmetric group of degree N. As we will see, the collective coordinate quantization on this moduli space will lead to a Hilbert space including non-BPS quantum states, while the quantum BPS states form only a subspace.

The collective coordinate quantization was originally developed for (topological and non-topological) solitons in scalar field theory. Classically the internal conserved observables of the solitons are generated by rotating the collective coordinates of a static solution in internal space. So to quantize the value of the internal observables, naturally one needs to turn the collective coordinates into quantum dynamical variables. In the present case, we promote the variables a and U in eq. (5.1) to dynamical variables:

$$a \to a(t), \qquad U \to U(t). \tag{5.3}$$

(Alternatively, we may absorb the exponential factor $e^{-it/R}$ into the diagonal part with $a(t) \to a(t)e^{it/R}$. We will not use this convention however.) Recall that eq. (5.1) is in the $A_0 = 0$ gauge. There is a residue global U(N) symmetry in this gauge, and because of the original color gauge symmetry in the SYM we need to impose the Gauss's law constraints.

Substituting eq. (5.1) with (5.3) into the original Lagrangian of SYM, we get the Lagrangian for the collective coordinates $a(t)$, $U(t)$:

$$L = \mathrm{Tr}\left\{\dot{a}\dot{a}^\dagger - \frac{i}{R}(a^\dagger \dot{a} - a\dot{a}^\dagger) - \frac{1}{2}[a,\omega][a^\dagger,\omega]\right\} \tag{5.4}$$

where $\omega := i\sqrt{2}\dot{U}U^\dagger$, $\omega^\dagger = \omega$. Because $U \in \mathrm{U}(N)/\,\mathrm{U}(1)^N$, $\omega^i_i = 0$ for $i = 1, 2, \ldots, N$. In terms of the matrix elements, the Lagrangian (5.4) reads

$$L = \sum_{i=1}^{N}\left(|\dot{a}_i|^2 - \frac{i}{R}(a_i^*\dot{a}_i - a_i\dot{a}_i^*) + \frac{1}{2}\sum_{i\neq j}|a_i - a_j|^2\omega^i_j\omega^j_i\right). \tag{5.5}$$

The first two terms in eq. (5.5) define a standard Landau problem for N particles, with cyclotron frequency $1/R$. The origin of the "magnetic field" is due to the rotation $e^{-it/R}$ in the R-ball solution that generates R-charge. The last term is in the standard form for a top rotating in a homogeneous space with symmetry G, with I_{AB} the *inertia tensor* and ω^A the *angular velocities* taking values in the Lie algebra of G. In the present case, the group G is U(N). The inertia tensor is diagonal with element I^{ii}_{jj} given by $|a_i - a_j|^2$. The canonical momenta are

$$J^i_j = \frac{\partial L}{\partial \omega^j_i} = \left\{ \begin{array}{ll} |a_i - a_j|^2 \omega^i_j, & i \neq j; \\ 0, & i = j \end{array} \right\}, \tag{5.6}$$

with the Poisson structure

$$\{J^i_j, J^k_l\}_{P.B.} = \delta^i_l J^k_j - \delta^k_j J^i_l \ . \tag{5.7}$$

Observe that

$$a_i \neq a_j, \quad \forall i \neq j \iff \det \left(\frac{\partial^2 L}{\partial \omega^i_j \partial \omega^k_l} \right) = |\Delta(a)|^2 \neq 0 \,, \tag{5.8}$$

where $\Delta(a)$ is the van DeMonde determinant for $(a_1, a_2, \ldots, a_N)$:

$$\Delta(a) = \prod_{i<j} (a_i - a_j). \tag{5.9}$$

So the Hessian for (5.6) is nonsingular and, therefore, the Hamiltonian is well-defined *only* in the subspace of moduli without coinciding eigenvalues. We will do canonical quantization on this subspace with the Hamiltonian

$$H = H_0 + \sum_{i \neq j} \frac{J^i_j J^j_i}{2|a_i - a_j|^2}, \tag{5.10}$$

where

$$H_0 = \sum_{i=1}^{N} \left(p_i + \frac{i a^*_i}{R} \right) \left(p^*_i - \frac{i a_i}{R} \right) \,, \tag{5.11}$$

and the canonical momenta are given by $p_i = \dot{a}^*_i - i a^*_i / R$, $p^*_i = \dot{a}_i + i a_i / R$. The Hamiltonian in (5.10) is a *generalized* Calogero-Sutherland model for U(N)-spin [25] coupled to a constant magnetic field. By "generalized", we mean that the variables a_i are complex instead of real numbers. This difference will dramatically change the physics at the quantum level.

To quantize the system, we first promote the Poisson brackets (5.7) to the commutation relations for $su(N)$ Lie algebra:

$$[J^i_j, J^k_l] = i(\delta^i_l J^k_j - \delta^k_j J^i_l) \ . \tag{5.12}$$

Classically the Gauss's law $[Z^\dagger, D_0 Z] + [Z, D_0 Z^\dagger] = 0$ on the moduli space reads

$$[a^\dagger, [a, \omega]] + [a, [a^\dagger, \omega]] = 0, \tag{5.13}$$

or equivalently, in terms of the U(N) angular momenta,

$$J^i_j = 0. \tag{5.14}$$

At the quantum level, the Gauss's law (5.14) is promoted to the constraints on the physical states:

$$J^i_j |phys\rangle = 0. \tag{5.15}$$

To see the meaning of the constraints, we introduce a coordinate representation: $|phys\rangle \to \psi(a, U)$. Then J^i_j are represented by the *right-invariant vector fields* on the U(N) group manifold that generate left translations:

$$J^i_j = -iU^i_k \frac{\partial}{\partial U^j_k}; \quad \left(1 + \epsilon^i_j J^j_i\right) f(U) = f\left((1 - i\epsilon)U\right). \tag{5.16}$$

The Lie algebra relation (5.12) is readily to verify. Moreover, it is obvious that $J^i_i \psi = 0$. The Gauss's law (5.15) is equivalent to $\psi(a, U) = \psi(a)$. Namely, the wavefunction of physical states are independent of the coordinates U. So we will consider only physical states in the form $\psi(a)$. Thus the physical degrees of freedom are reduced to the diagonal elements a_i, giving rise to the many-body interpretation of the quantum states. As we have seen from (5.8), a Hamiltonian formalism is well defined only on a subspace of the moduli with a_i all unequal. The reduced moduli space for a_i's is then $\{\mathbf{C}^N - D\}/S_N$, where D is the set of points in $\mathbf{C}^N$ with coinciding coordinates. The fundamental group of this reduced moduli space is known to be the braid group of N-particles that classifies the quantum statistics in two dimensions [26]. This is the origin of the emergence of non-trivial statistics, including fermions, after quantizing scalar (bosonic) field configurations.

Finally, we consider the Hamiltonian in the subspace of physical states. There is a nontrivial measure in defining the inner product in the physical Hilbert space. This measure can be viewed as the Faddeev-Popov measure due to gauge fixing. In fact, in the space of all normal matrices, the measure is $dAdA^\dagger = d\mu(U) da da^\dagger |\Delta(a)|^2$, where $d\mu(U)$ is the descended Haar measure at point U on the coset space $\mathrm{U}(N)/\mathrm{U}(1)^N$. By integrating out the unphysical "angular part" $d\mu(U)$, the inner product of two physical states is given by

$$\langle\phi|\psi\rangle = \int \prod_i da_i da_i^\dagger |\Delta(a)|^2 \phi(a)^* \psi(a). \tag{5.17}$$

Then the Hamiltonian acting on the wavefunction ψ is identified with H_0 in (5.11) with the measure factor taking into account:

$$H = \sum_{i=1}^N \frac{1}{|\Delta|^2} \left(p_i + \frac{ia_i^*}{R}\right) |\Delta|^2 \left(p_i^\dagger - \frac{ia_i}{R}\right), \tag{5.18}$$

in which

$$p_i = -i\frac{\partial}{\partial a_i}, \qquad p_i^\dagger = -i\frac{\partial}{\partial a_i^*}. \tag{5.19}$$

A statistical interaction appears in H because of the nontrivial measure factor $|\Delta(a)|^2$. Similar to the one-dimensional case, this statistical interaction can be absorbed by a redefinition of wavefunction:

$$\psi(a) \to \Psi(a) := \Delta(a)\psi(a) \ . \tag{5.20}$$

The Hamiltonian acting on the wavefunction Ψ is given by

$$\mathbf{H} = \sum_{i=1}^{N} \frac{1}{\Delta^*}\left(p_i + \frac{ia_i^*}{R}\right)\Delta^*\Delta\left(p_i^\dagger - \frac{ia_i}{R}\right)\frac{1}{\Delta} = \sum_{i=1}^{N}\left(p_i + \frac{ia_i^*}{R}\right)\left(p_i^\dagger - \frac{ia_i}{R}\right), \tag{5.21}$$

which has the same form of H_0 due to the facts that $[p_i^\dagger, \Delta] = 0$, $[p_i, \Delta^*] = 0$. In deriving the second equality, we have explored the holomorphy of the factor Δ. The transformation (5.20) has the effect of attaching a statistical flux [27] to each particle in two dimensions, to turn the original bosons into fermions [26]. So the Hamiltonian (5.21) describes a free fermion system in a magnetic field. (Note that the above treatment is a bit more sophisticated than that of Berenstein's hermitian matrix toy model [2], because our Z is not hermitian.)

The ground states of the Hamiltonian (5.18) are determined by the following first-order equations

$$\left(\frac{\partial}{\partial a_i^*} + \frac{a_i}{R}\right)\psi = 0, \qquad i = 1, 2, \ldots, N. \tag{5.22}$$

For any i, the solution to eq. (5.22) is a lowest Landau level (LLL): $\phi_m(a_i) = a_i^m e^{-|a_i|^2}/\sqrt{m!}$ for $m = 0, 1, \ldots$; and the many-body ground states are the symmetrization of the LLL's for all i

$$\psi_G = \mathcal{C}_N \sum_{\text{sym}} \prod_i \phi_{m_i}(a_i), \tag{5.23}$$

where $\mathcal{C}_N$ is the normalization factor. Accordingly, the ground states for (5.21) are given by

$$\Psi_G = \Delta(a)\psi_G. \tag{5.24}$$

Thus the quantum half BPS states are identified with the ground states, which are infinitely degenerate. Excited states come from higher Landau levels, and generically are non-BPS. (Though the many-body system is a free system, there are statistical correlations coming from the measure.) The gap between the LLL and excited states is of order $1/R$. So in the small-R limit, the quantum states will be projected down to the LLL, i.e. the half BPS sector.

5.2 Canonical quantization

In this subsection, we are going to show that a direct application of canonical quantization to the matrix elements of Z for the commutative half BPS R-balls can reproduce the results of collective coordinate quantization. Recall the classical system to be quantized:

$$\dot{Z} = iZ, \ \ [Z, Z^\dagger] = 0, \tag{5.25}$$

where we have set $R = 1$ so that all quantities are dimensionless. The canonical momenta are identified to be $P = \dot{Z}^\dagger$ for this first-order system. We promote each matrix element of Z to an operator, and impose the canonical commutation relations $[Z^i_j, P^k_l] = i\delta^i_l\delta^k_j$. Using the first equation in (5.25), we have

$$[Z^i_j, Z^{\dagger k}_l] = -\delta^i_l\delta^k_j. \tag{5.26}$$

The second equation in (5.25) now can not hold for operators; so we impose it as constraints on physical states. More explicitly, we introduce

$$L^i_j := Z^i_k Z^{\dagger k}_j - Z^k_j Z^{\dagger i}_k. \tag{5.27}$$

Note that they are automatically traceless: $\sum_i L^i_i = 0$. It is easy to check that they generate the $su(N)$ Lie algebra:

$$[L^i_j, L^k_l] = \delta^k_j L^i_l - \delta^i_l L^k_j. \tag{5.28}$$

In other words, L^i_j provide the Schwinger oscillator representation of $su(N)$. In this formalism, the second equation in (5.25) is promoted to the constraints

$$L^i_j|phys\rangle = 0, \tag{5.29}$$

so the quantum states are $\mathrm{SU}(N)$ singlets. For these states, Gauss's law

$$([Z, \dot{Z}^\dagger] + [Z^\dagger, \dot{Z}])|phys\rangle = 0,$$

and the BPS conditions $(\Delta - Q)|BPS\rangle = 0$ are automatically satisfied because of eqs. (5.25), with

$$\Delta = Q = \mathrm{Tr} Z Z^\dagger. \tag{5.30}$$

In this scheme, all the physical states saturate the BPS bound $\Delta = Q$ quantum-mechanically, since we started with a first-order system, equivalent to a LLL system.

The wavefunction of quantum states can be defined in the coherent-state (complex coordinate) representation by $|phys\rangle \to \Psi(Z)$ and $Z^\dagger \to \partial/\partial Z^T$. (Here the superscript T stands for matrix transpose.) In fact, this is a Bargmann-Fock representation in the space of holomorphic functions $\Psi(Z)$ with measure $d\mu(Z, Z^\dagger) = e^{-\mathrm{Tr}ZZ^\dagger}$. Recall that $Z\partial/\partial Z^T$ generates the left $\mathrm{U}(N)$ action on Z, while $Z^T\partial/\partial Z$ the right action:

$$\mathrm{Tr}\left(\epsilon Z \frac{\partial}{\partial Z^T}\right)\Psi(Z) = \Psi((1+\epsilon)Z) - \Psi(Z), \qquad \mathrm{Tr}\left(\epsilon Z^T \frac{\partial}{\partial Z}\right)\Psi(Z) = \Psi(Z(1+\epsilon)) - \Psi(Z). \tag{5.31}$$

So L^i_j generate *similar* transformations in $\mathrm{SL}(N, \mathbf{C})$ by $\Psi(Z) \to \Psi(gZg^{-1})$ in the complex domain and the Gauss's law dictates that

$$\Psi(gZg^{-1}) = \Psi(Z), \ \forall g \in \mathrm{SL}(N, \mathbf{C}). \tag{5.32}$$

Note that this $\mathrm{SL}(N, \mathbf{C})$ is not a symmetry on the Hilbert space, but a symmetry on the wavefunctions for physical states. Eqs. (5.32) and (5.30) are the major results in the literature on 1/2 BPS states.

5.3 Quantum Hall analogy and holography

In the above we have obtained the wavefunction (5.24) for half BPS quantum states. It describes a many-body system of N particles, which we call G-particles. Here the name G hints about two features of these particles: their origin in the gauge (color) degrees of freedom (their number N being the rank of the gauge group $\mathrm{U}(N)$) in SYM and their close relation to *geometry* or *gravity* in the holographic dual (see below).

The simplest ground state with the wavefunction (5.24) corresponds to $m_i = 0$ for all i. It has the minimal angular momentum or R-charge $N^2/2$. In this case, the wavefunction is nothing but the Laughlin wavefunction for a quantum Hall droplet with filling factor $\nu = 1$, an incompressible quantum fluid forming a circular disk. (For an introduction of the QHE for particle physicists see, e.g., [28]; for that for string theorists see, e.g., [13, 10].) The general states described by a wavefunction (5.23) represent planar fluid composed of discontinuous components of the form of concentric rings. If the single particle states have larger enough angular momenta, they form a free 2D fermion gas in the LLL.

Compared with Berenstein's treatment, our R-ball approach has the advantage that one can see clearly the origin of the emergence of "Landau levels", on which the quantum Hall analogy is based. Essentially this is due to the rotation (the time-dependent factor $\exp(-it/R)$ in eq. (5.1)) of the R-balls that generates R-charge. By now it is well-known that the rotation of a BEC will lead to the emergence of an effective magnetic field in the co-moving frame and of the single-particle Landau levels [29]. Numerically it has been shown that with small filling fractions, even fermionic QH-like states [30], including FQH-like states [31], should appear in rotating BEC's. Actually the LLL states (but not QHE yet) in a rotating BEC has been seen experimentally [32]. What we have seen above in the half BPS sector in SYM is essentially the same physics: An R-ball in $\mathcal{N} = 4$ SYM is nothing but a rotating BEC; and the small radius limit ($R \to 0$ corresponds to rapidly rotating BEC, which is indeed the lowest Landau level regime in the real atomic BEC experiments.)

To understand the solutions (4.6) from the point of view of dual IIB superstring theory, we claim that the G-particles satisfying Fermi statistics correspond to the LLM fermions that are the "sources" of the half BPS geometry in a large N limit[1] in the LLM's construction [4]. The state with minimal R-charge (for fixed N) is a circular droplet with a uniform distribution of the LLM fermions, which is known to be the "source" of the $AdS_5 \times S^5$ geometry. The states whose R-charge are not far above $N^2/2$ can be viewed as a few G-particles excited a bit outside the droplet, corresponding to a few giant graviton excitations in AdS_5. The general discontinuous fluid states of G-particles that form concentric rings correspond to general LLM's half BPS geometries seeded by concentric ring-like distribution of LLM fermions. Finally, generically not every possible state of the G-particle gas correspond to a classical geometry.

[1]If N is not large enough, in general we do not have classical geometry in dual string theory.

6. Quantization of non-commutative $1/4$ BPS sector

The success in the last section in making the connection of half BPS R-balls to the quantum Hall effect meaningful substantially encourages us to proceed to examine the non-commutative quarter BPS R-balls and to confirm the appearance of FQH-like states after quantization.

6.1 Collective coordinate quantization

Similar to the half BPS case, we first try to quantize the classical R-balls (4.17) with collective coordinate quantization. From the analysis of the moduli space in section 4, we can parametrize this type of R-ball solutions in the following way:

$$Z_\alpha = \sqrt{2\theta} e^{i\varphi_\alpha} a_\alpha^V, \quad \alpha = 1, 2, \tag{6.1}$$

where $a_\alpha^V = V^\dagger a_\alpha V$ contains the gauge degrees of freedom labeled by V in the coset space $\mathrm{SU}(\infty)/\mathrm{U}(2)$ and φ_α are two independent phases. (Here we have absorbed a factor $\exp(\pm it/R)$ into $\varphi_{1,2}$ respectively.) Subsequently, the physical and dynamical degrees of freedom are $\sqrt{\theta}$, φ_α, which span a reduced moduli space $\mathbf{R}^+ \times \mathrm{U}(1) \times \mathrm{U}(1)$ for the non-commutative 1/4 BPS sector.

Now we measure the length in units of the radius R of S^3; in the large N limit defined in section 3, we introduce $r := 2\sqrt{c_2}$, with $c_2 = N(N_1 + N_2)\theta/R^2$. We also introduce the following notions:

$$\frac{N_2}{N_1} = \frac{q}{p},$$

where p and q are nonnegative and coprime. Then the Lagrangian is given by

$$L = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\left(\frac{p}{p+q}\dot{\varphi}_1^2 + \frac{q}{p+q}\dot{\varphi}_1^2\right) - \frac{r^2}{2}. \tag{6.2}$$

Gauss's law is reduced to

$$G := r^2(\dot{\varphi}_1 + \dot{\varphi}_2) = 0. \tag{6.3}$$

By the standard procedure, one obtains

$$H = \frac{p_r^2}{2} + \frac{1}{2r^2}\left(\frac{p+q}{p}J_1^2 + \frac{p+q}{q}J_2^2\right) + \frac{r^2}{2}, \tag{6.4}$$

$$Q = J_1 - J_2, \tag{6.5}$$

$$G = (p+q)\left(\frac{J_1}{p} + \frac{J_2}{q}\right) \tag{6.6}$$

where $p_r = \dot{r}$, $J_1 = pr^2\dot{\varphi}_1/(p+q)$ $J_2 = qr^2\dot{\varphi}_2/(p+q)$.

Upon quantization, $J_\alpha = -i\partial/\partial\varphi_\alpha \to m_\alpha$, $(\alpha = 1, 2)$, with m_α integers. It is easy to see the relation of J_α and the Schwinger representation of the $u(2)$ algebra: $\mathbf{L}_0 = J_1 + J_2$, $\mathbf{L}_3 = J_1 - J_2$. Then Gauss's law dictates that

$$qm_1 + pm_2 = 0 \;\Rightarrow\; m_1 = pk,\; m_2 = -qk, \tag{6.7}$$

where k is an integer. Accordingly, in the physical subspace

$$\begin{aligned} H &= -\frac{1}{2r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(p+q)^2k^2}{2r^2} + \frac{r^2}{2}, \\ Q &= (p+q)k. \end{aligned} \tag{6.8}$$

The eigenstates of the Hamiltonian in (6.8) of energy E are given by

$$\Psi_{nk}(r,\varphi_1,\varphi_2) = r^{(p+q)|k|}e^{-r^2/2}F(-n,(p+q)|k|+1,r^2)e^{ik(p\varphi_1-q\varphi_2)} \tag{6.9}$$

where $F(.,.,r^2)$ is a confluent hypergeometric function and $n = (E-1-(p+q)|k|)/2 = 0,1,2,\ldots$. Then the charge density and energy density are given by

$$Q = (p+q)k, \qquad E = (p+q)|k| + 2n + 1. \tag{6.10}$$

Ignoring the zero-point energy from the ordering ambiguity of quantum operators, the BPS states are those corresponding to $n = 0$.

Moreover, it is obvious that the system (6.8) can be mapped to a two-dimensional harmonic oscillator in terms of the new variables:

$$\varphi = \frac{p\varphi_1 - q\varphi_2}{p+q}, \qquad z = re^{i\varphi}. \tag{6.11}$$

Recall that p and q are coprime if and only if there are two integers s, t such that $pt-qs = 1$ (Bézout's identity; it is easy to see that s, t are also coprime). Then the period of φ is actually $2\pi/(p+q)$, corresponding to $\varphi_1 \to \varphi_1 + 2t\pi$, $\varphi_2 \to \varphi_2 + 2s\pi$. Consequently, the two-dimensional harmonic oscillator is actually defined on a cone. The spectrum of $E-Q$ from (6.10) can be mapped to that of the Landau levels labelled by n with $k = 0,1,2,\ldots$ labelling the degeneracy in the same Landau level (and n-th "anti-Landau level" for $k = 0,-1,-2,\ldots$).

We know that an $\mathrm{SL}(2,\mathbb{Z})$ transformation changes a pair of coprime integers into another coprime pair; and any coprime pair (p,q) can always be generated by acting an element of $\mathrm{SL}(2,\mathbb{Z})$

$$\begin{pmatrix} p & s \\ q & t \end{pmatrix}$$

on a standard vector $(1,0)^T$. Since the coprime pair is determined by the ratio N_1/N_2, the different choices for the coprime pair correspond to different large N sectors in $\mathcal{N}=4$ SYM, and $\mathrm{SL}(2,\mathbb{Z})$ transforms between different sectors with the same quantized k.

6.2 Canonical quantization

According to eq. (6.10), the quantum quarter BPS state obtained here contains essentially one quantum number k, hinting that the (many-body) state is a rigid or incompressible one. This picture is going to be checked in this subsection by applying another quantization scheme. In contrast to the above quantization scheme, in which the classical Gauss's law constraints except one are solved before quantization, now let us start from the classical solutions (4.11) with (4.17), treat $a_\alpha(t)$ as dynamical variables and apply canonical quantization to them, with the non-commutativity constraints

$$[a_\alpha(t), a_\alpha^\dagger(t)] = 2\theta \mathbf{1}_{N_\alpha \times N_\alpha}, \qquad (\alpha = 1, 2), \tag{6.12}$$

incorporated by Langrangian multipliers. This results in the effective Lagrangian (with two matrix Lagrangian multipliers λ_α):

$$\begin{aligned} L = & \frac{N_2}{2R}\mathrm{Tr}(\dot{a}_1\dot{a}_1^\dagger) + \frac{N_1}{2R}\mathrm{Tr}(\dot{a}_2\dot{a}_2^\dagger) \\ & + \frac{iN_2}{2R^2}\mathrm{Tr}(a_1 da_1^\dagger - da_1 a_1^\dagger) - \frac{iN_1}{2R^2}\mathrm{Tr}(a_2 da_2^\dagger - da_2 a_2^\dagger) \\ & + \frac{2\theta N_2}{R^2}\mathrm{Tr}\lambda_1 - \frac{2\theta N_1}{R^2}\mathrm{Tr}\lambda_2, \end{aligned} \tag{6.13}$$

where $da_\alpha = \dot{a}_\alpha - i[\lambda_\alpha, a_\alpha]$. The direct product structure of $Z_\alpha(t)$ ensures that the composite operators formed by Z_α in SYM do not receive loop corrections to their conformal dimensions. In the $R \to 0$ limit we can drop the kinetic term in the Lagrangian (6.13), as long as we focus on the ground states. Finally we end up with the effective matrix model:

$$\begin{aligned} L &= L_1 - L_2, \\ L_1 &= \frac{iN_2}{2R^2}\mathrm{Tr}(a_1 da_1^\dagger - da_1 a_1^\dagger) + \frac{2N_2\theta}{R^2}\mathrm{Tr}\lambda_1 \\ L_2 &= \frac{iN_1}{2R^2}\mathrm{Tr}(da_2 a_2^\dagger - a_2 da_2^\dagger) + \frac{2N_1\theta}{R^2}\mathrm{Tr}\lambda_2. \end{aligned} \tag{6.14}$$

We see that L_1 or L_2 is separately a NCCSMM model that has been discussed by Susskind [10]:

$$L = \frac{i\xi}{2R^2}\mathrm{Tr}(udu^\dagger - duu^\dagger) + \frac{2\xi\theta}{R^2}\mathrm{Tr}\lambda, \tag{6.15}$$

where $\xi = N_2$ or N_1, depending on whether $u = a_1$ or $u = a_2^\dagger$. This NCCSMM model has a different origin, compared with the matrix model in ref. [14], where a chemical potential μ was introduced in $\mathcal{N} = 4$ SYM as an external parameter.

To quantize, we introduce the operators $\mathbf{x}_{ij}$ and $\mathbf{y}_{ij}$ through $\mathbf{u}_{ij} = \mathbf{x}_{ij} + i\mathbf{y}_{ij}$, and impose the canonical commutation relations:

$$[\mathbf{x}_{ij}, \mathbf{y}_{mn}] = i\frac{R^2}{N\xi}\delta_{in}\delta_{jm}. \tag{6.16}$$

Here we use the fact that N^{-1} plays the role of the Planck constant in a large N matrix model. The non-commutative constraints are now imposed on the physical states:

$$(\mathbf{x}_{ij}\mathbf{p}_{jm} - \mathbf{p}_{ij}\mathbf{x}_{jm})|\Psi\rangle = \frac{i}{\nu}\delta_{im}|\Psi\rangle, \tag{6.17}$$

where $\nu = R^2/N\xi\theta$ and $\mathbf{p}_{ij} = \xi\mathbf{y}_{ij}/R^2$.

The left hand side of (6.17) resembles the angular momentum operator in quantum mechanics, which generates a rotation of particles in a two-dimensional plane. We may define a unitary operator to generate such a rotation:

$$\hat{T} = \exp\{\omega_{im}(\mathbf{x}_{ij}\mathbf{p}_{jm} - \mathbf{p}_{ij}\mathbf{x}_{jm})\}, \tag{6.18}$$

with ω_{im} the angles of rotation. As we consider the operation to exchange two particles, i.e., to rotate them by an angle $\mathrm{Tr}\omega = \pi$, we have

$$\hat{T}|\Psi\rangle = e^{i\pi/\nu}|\Psi\rangle. \tag{6.19}$$

This indicates that the many-body state $|\Psi\rangle$ is a QH state of fermions when $1/\nu$ is odd, or of bosons when $1/\nu$ is even. So $1/\nu$ is just the filling fraction of the QH system. The well-known quantization of ν in the NCCSMM model implies $\theta = k/N\xi R^2$, with k a positive integer.

Applying the above results to our matrix model (6.14), we have

$$\left.\begin{aligned} \nu_1^{-1} &= k_1 = NN_2\theta/R^2, \\ \nu_2^{-1} &= k_2 = NN_1\theta/R^2, \end{aligned}\right\} \quad \Rightarrow \quad k_1 = qk,\ k_2 = pk, \tag{6.20}$$

where k is an integer, and (p, q) is again a coprime pair defined by the ratio $N_2/N_1 = q/p$. Substituting the quantized θ into eq. (4.21), we obtain the quantized R-charge

$$Q = c_2 = (p + q)k. \tag{6.21}$$

It is the same as we obtained before from the collective coordinate quantization.

6.3 New higher dimensional quantum Hall state

In the above we have constructed mathematically a new quantum BPS state in $\mathcal{N} = 4$ SYM, with energy equal to its R-charge. What is the physical interpretation of this state? It is known that the NCCSMM model (6.15) describes a FQH system with filling factor $\nu = 1/k$. So the ground state of our model (6.14) describes a quantum state that is the product of the two FQH states, respectively, on two orthogonal non-commutative planes. To get some ideas about what it looks like, we note that for the classical R-ball solution (4.17), the constant matrices A_1 and A_2 are neither Hermitian nor normal. However, one may form two Hermitian matrices from them: $Z_1Z_1^\dagger = A_1A_1^\dagger$ and $Z_2Z_2^\dagger = A_2A_2^\dagger$, which can be diagonalized simultaneously by a unitary rotation, since one can easily verify $[A_1A_1^\dagger, A_2A_2^\dagger] = 0$. It is not hard to see that there are only N_1 independent eigenvalues of $A_1A_1^\dagger$, $|a_{1,i}|^2$, $(i = 1, 2, \ldots, N_1)$, and N_2 independent eigenvalues of $A_2A_2^\dagger$, $|a_{2,j}|^2$, $(j = 1, 2, \ldots, N_2)$. Upon quantization, $|a_{1,i}|^2$ and $|a_{2,j}|^2$ can be interpreted, respectively, as the radial positions (squared) of G-particles in Z_1- and Z_2-plane. Here as in the commutative half BPS case, we adopt the interpretation of matrix diagonal elements as coordinates of particles, which we have named as G-particles, in the same spirit as the BFSS matrix model [35]. Note that Z_1 and Z_2 are N-by-N matrices with $N = N_1N_2$, which is just the number of G-particles. So the distribution of N G-particles in four-dimensional

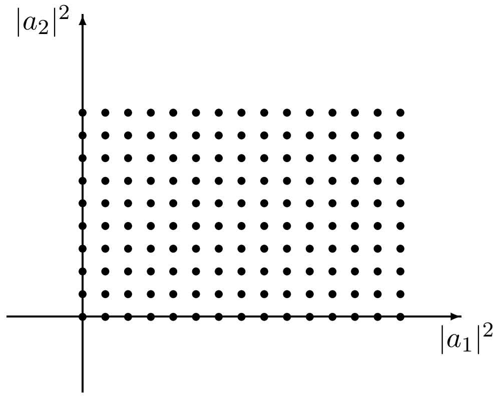

Figure 1: The distribution of N G-particles in ($|a_1|^2$, $|a_2|^2$) space. The lattice spacing is θ.

internal space thus forms a rectangular lattice with spacing θ in the two-dimensional plane spanned by $|a_1|^2$ and $|a_2|^2$ (figure 1). Clearly there are N_1 columns of G-particles (figure 1) distributing along $|a_1|^2$ direction, and each of them has N_2 rows of G-particles (figure 1) along $|a_2|^2$-direction. Altogether there are $N_1 N_2$ G-particles distributed on a four dimensional space, that is the product of two circular disks, respectively, on non-commutative Z_1- and Z_2-plane. Each column (or row) represents a quantum Hall droplet. So their direct product represents a higher dimensional quantum Hall state in four dimensions. (It is shown that the Landau Hamiltonian in flat space with even dimensions can be reduced to the direct sum of two dimensional Landau Hamiltonians [34].)

What would be the holographic correspondence of the new 1/4 BPS states given by eq. (6.10) or (6.21) in the IIB string dual? The above figure motivates us to suggest the following picture: The G-particles, thought of as "sources" in IIB supergravity like LLM particles in the LLM construction, form a four dimensional object: namely we have a bunch (N_2) of FQH droplets, each living on the Z_1-plane, consisting of N_1 G-particles and looking like a point-like object on the Z_2-plane; and the bunch of N_2 point-like objects also form a FQH droplet on the Z_2-plane. This state is certainly not the FQH state of LLM fermions that the present authors suggested in the IIB quantum gravity about a year ago [9], which is known as a deformation of the 1/2 BPS IIB geometry to have null singularity. The above picture for the states (6.10) immediately suggests themselves as a resolution of our previously proposed FQH states of LLM fermions: Namely in the limit $N_1, N_2 \to \infty$ with fixed $N_2/N_1 = q/p \ll 0$, the quantum states (6.10) become a candidate for the SYM dual of the FQH states in IIB gravity suggested by us [9]. Indeed, we can present several evidences for this suggestion:

- We may impose an extra condition, $g^2 N_2 \sim$ fixed, According to the standard AdS/CFT dictionary, the typical length scale along Z_1-plane is $l_s(g^2 N_1)^{1/4} \gg l_s$ with l_s the stringy scale, while the typical scale along Z_2 plane is $l_s(g^2 N_2)^{1/4} \sim l_s$. Therefore, the classical geometry along Z_1-plane is well-defined, and we can identify this plane as the boundary plane of LLM geometries. Meanwhile, the classical ge-

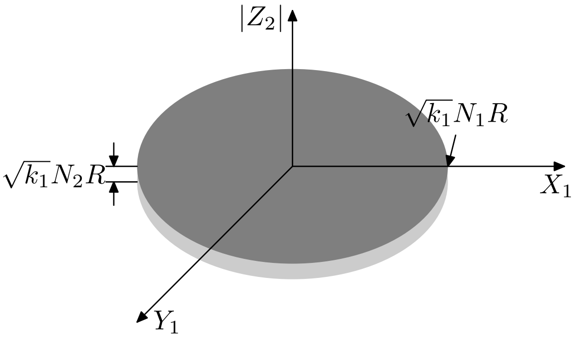

Figure 2: The semi-classical configuration of a four dimensional fractional quantum Hall state. $|Z_2|$ denotes the radius in the transverse 2-plane.

ometry description along Z_2-plane breaks down, and the quantum corrections play a role at the string scale and resolve the original singularity.

- The angular momentum or R-charge contributed by Z_1 and Z_2 are proportional to pkN_1^2 and pkN_1N_2, respectively. The area occupied by a FQH droplet is proportional to its angular momentum. Hence the typical size of the quantum states is N_1R along Z_1-plane and $\sqrt{N_1N_2}R$ along Z_2-plane. If we take $N_1R \sim$ fixed as a macroscopic scale, $\sqrt{N_1N_2}R \to 0$ will be a microscopic scale. Then the configuration (4.11) looks like a thin pancake in internal four-dimensional space (figure 2), and the states looks like a two-dimensional incompressible fluid (FQH fluid) macroscopically. The thin thickness of such FQH fluid in the transverse directions can be understood as a necessity for the resolution of null singularity in dual string theory.

- The Hamiltonian H or R-charge is a simple summation, $H = H_1 + H_2$, where H_1 and H_2 denoted the contribution from Z_1 and Z_2 respectively. Since $Q_2/Q_1 = q/p \ll 0$, H_2 can be treated as a perturbation. At the zeroth order, we have only H_1, which describes a 2d FQH system, as what we have suggested in ref. [9].

Of course, our result showed that this resolution actually breaks more supersymmetries. Because $\mathcal{N} = 4$ SYM is a well-defined quantum theory that is believed to contain complete information on IIB superstring theory, our study suggests a possible way to deal with the properties of the spacetime geometries near null singularities, or emergent geometry [37], in terms of dual quantum field theory or its reductions to matrix models.

To conclude this subsection, we make two remarks: First, it would be interesting to see whether a smooth IIB geometry could be generated by such "seeds" in the IIB supergravity dual. We note that the solution (4.11) and the above quantization procedure preserve the isometry group $\mathrm{SO}(2) \times \mathrm{SO}(2) \times R$. Second, the statistics we were talking about in the last subsection is the statistics of N_1 G-particles in one fixed FQH droplet on Z_1-plane and the statistics of the N_2 FQH droplets as identical objects when two of them are exchanged on Z_2-plane. Because of the direct product structure of the total quantum state, interchanging any two of N_1N_2 G-particles is not an admissible symmetry operation.

7. Summary and discussions

A central issue for understanding AdS/CFT holography is to see how geometry or gravity emerges in the CFT dual. In particular, one wants very much to see how the LLM fermions in a quantum Hall droplet, that are known [4] to “encode” a wide class of half BPS IIB geometries, arise in the dual gauge theory. In this paper we have proposed a new framework for constructing quantum candidate states in $\mathcal{N} = 4$ SYM on $R \times S^3$, which are promising for holographically encoding classical or quantum geometries on the gravity side.

In our proposal, these candidates are *quantized* R-ball states, with energy saturated by a conserved U(1) R-charge. They are constructed by quantization over the moduli space of certain classical R-balls, which are spatially constant, time-dependent (rotating in internal space) and maintain a fraction of supersymmetries. Many features of the Berenstein's matrix model for the commutative half BPS sector emerge naturally in our framework with space-filling R-balls. In particular, the origin of the “magnetic field” in the QH analogy is identified to be the rotation of the R-balls in internal space that generates R-charge, and the origin of the projection down to the lowest Landau level is closely related to the BPS bound (the energy is saturated by the R-charge). Quantization of such R-balls results in a many-body quantum Hall system with filling factor $\nu = 1$, whose constituents can be identified with the LLM fermions. The system is a non-interacting one, whose constituents are called G-particles with their number related to the rank N of the color gauge group. Gauge invariance (the Gauss's law) plays an important role in reducing the number of physical degrees of freedom. In the half BPS case, this reduces the degrees of freedom in the physical quantum states from N^2 to N.

The success in making the QH analogy of the half BPS dynamics meaningful and substantial encouraged us to look for FQH-like states in the quarter BPS sector. In our framework, we have been able to shown that non-commutative almost BPS classical R-balls are allowed in the large N limit, with two non-vanishing complex scalars. Upon quantization they lead to a NCCSMM model that describe the “direct product” of two QH droplets on a pair of orthogonal planes, each in $\nu = 1/k$ FQH states (k being an integer). Thus the quantum states are those of an interacting many-body system, actually a new four-dimensional QH system. In a special limit, the states reduces approximately to the FQH states on a plane that correspond to the incompressible giant graviton fluid (with density $\rho = 1/k$ proposed by the present authors [9] previously in the dual gravity theory). The latter was known to give rise to geometry with null singularities, and we interpret the four dimensional new QH states obtained as representing a resolution of the null singularities in the quantum theory of gravity. (Note that the $d = 4$ QH effect we have here is *not* as the $d = 4$ QHE proposed previously in ref. [38, 39], which are not a direct product of two Abelian QHE.)

We note that both the compactness of the space S^3 and non-commutativity permitted in the large N limit with $N \to \infty$ play an essential role in admitting the existence of the new (FQH-like) R-ball states. First, the conformal coupling term, that couples the scalars to the spatial curvature, gives rise to a harmonic confining potential over the usual moduli space of vacua in Minkowski spacetime. Second, S^3 has a finite volume. Combining these two facts, it makes sense to consider space-filling R-balls with a rotation in internal space generating an R-charge and to examine the small radius $R \to 0$ limit. This limit allows us not only to single out the lowest Kaluza-Klein modes, but also to project the R-balls down to LLL. This is because with a particular rotation frequency, the centrifugal force just cancels the harmonic confining potential, leading to the LLL (or BPS) states. Finally, we found the existence of the non-commutative R-balls that, upon quantization, exhibit FQH-like behavior and become BPS only in the large N limit, which is just the defining limit for AdS/CFT holography. This suggests to us that non-commutative geometry should play a profound role in studying the emergent gravity in the holographical CFT dual.

On the physics side, conceptually our study has heavily explored the analogy with two recent inter-related developments (BEC and QHE) in many-body systems. This is not surprising, since string/M theory essentially is a many-body system from the point of view of the BFSS matrix model [35]. The present authors hold the belief that the string/M theory has a profound connection with strongly correlated systems that are one of the recent focuses of attention in quantum many-body physics. One important concept is that of BEC, which plays a crucial role in the present context: the R-balls can be viewed as a rotating BEC on S^3 and what we have studied is the dynamics of a rapidly rotating BEC. The rotating BEC is related to the QH effect for fermions, both integral and fractional, at small boson filling fractions [31]. Previously a possible realization of the QHE in string theory has been proposed in the literature [40], which involved particular configurations of certain branes. Our present study suggests a more fundamental and ubiquitous connection of string theory with the QHE in particular and with non-commutative geometry in general.

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A. $\mathcal{N} = 4$ supersymmetry algebra on $R \times S^3$

By a straightforward calculation, the supersymmetry transformation (2.8) leads to the following supercurrents:

$$\begin{aligned}
\bar{J}_L^\mu &= -\frac{i}{2}\mathrm{Tr}(\bar{\psi}F_{\rho\sigma})\gamma^\mu\gamma^{\rho\sigma} - \mathrm{Tr}\{\bar{\psi}(\alpha^i D_\nu X_i + i\gamma_5\beta^j D_\nu Y_j)\}\gamma^\mu\gamma^\nu \\
&\quad + \frac{1}{2}\epsilon^{ijk}\mathrm{Tr}\{\bar{\psi}(\alpha^k[X_i, X_j] + \beta^k[Y_i, Y_j])\}\gamma^\mu - i\mathrm{Tr}(\bar{\psi}[X_i, Y_j])\alpha^i\beta^j\gamma^\mu\gamma_5 \\
&\quad - \frac{i}{R}\mathrm{Tr}\{\bar{\psi}(X_i + i\gamma_5\beta^j Y_j)\}\gamma^\mu\gamma_5\gamma^0, \\
\bar{J}_R^\mu &= \bar{J}_L^\mu + \frac{2i}{R}\mathrm{Tr}\{\bar{\psi}(X_i + i\gamma_5\beta^j Y_j)\}\gamma^\mu\gamma_5\gamma^0.
\end{aligned} \tag{A.1}$$

In a curved space with constant curvature, global supercharges associated to the above supercurrents can be defined by appropriately projecting the locally-defined $\bar{J}^0$ to a global section. To this end, we introduce the transformation

$$\zeta_L = M_L \zeta_{L0}, \qquad \zeta_R = M_R \zeta_{R0}, \tag{A.2}$$

where the 4×4 matrices $M_{L,R}$ depend on spacetime coordinates and ζ_{L0} and ζ_{R0} are two constant Majorana spinors. Then the global supercharges can defined with the help of ζ_{L0} and ζ_{R0}:

$$\bar{Q}_L = \int_{S^3} \bar{J}^0_L M_L, \qquad \bar{Q}_R = \int_{S^3} \bar{J}^0_R M_R. \tag{A.3}$$

We take the metric of $R \times S^3$ to be

$$ds_2^2 = dt^2 - d\theta^2 - \sin^2\theta(d\psi^2 + \sin^2\psi d\chi^2).$$

Accordingly, the explicit solution of the conformal Killing spinor equation (2.5) is given by [36, 16],

$$\epsilon = e^{\frac{it}{2R}\Gamma_0} e^{\frac{i\theta}{2R}\Gamma_{15}} e^{-\frac{\psi}{2R}\Gamma_{12}} e^{-\frac{\chi}{2R}\Gamma_{23}} \epsilon_0, \tag{A.4}$$

where ϵ_0 is a constant spinor, and Γ^a denotes γ-matrices in the local Lorentzian frame of AdS_5. Eq. (A.4) together with eq. (2.7) lead to

$$\begin{aligned} M_L &= e^{it/R} e^{-\frac{i\theta}{2R}\gamma_{01}} e^{-\frac{\psi}{2R}\gamma_{12}} e^{-\frac{\chi}{2R}\gamma_{23}}, \\ M_R &= e^{-it/R} e^{\frac{i\theta}{2R}\gamma_{01}} e^{-\frac{\psi}{2R}\gamma_{12}} e^{-\frac{\chi}{2R}\gamma_{23}}. \end{aligned} \tag{A.5}$$

Here γ_{ab}, as we noted in section 2, are defined in the local Lorentzian frame on $R \times S^3$.

We will focus on the fermionic part of the superconformal algebra that involves only the charges of R-ball configurations. In the $A_0 = 0$ gauge the variation of the supercurrent can be written as follows:

$$\begin{aligned} \delta_L \bar{J}^0_L &= -2iT^{0\nu}\bar{\zeta}_L\gamma_\nu - \frac{2i}{R}\bar{\zeta}_L\gamma^0\alpha^i\beta^j \mathrm{Tr}(X_i\dot{Y}_j - \dot{X}_iY_j) \\ &\quad + \frac{2}{R}\epsilon^{ijk}\bar{\zeta}_L\gamma^0\gamma_5(\alpha^k \mathrm{Tr}X_i\dot{X}_j + \beta^k \mathrm{Tr}Y_i\dot{Y}_j) + \cdots, \\ \delta_R \bar{J}^0_R &= -2iT^{0\nu}\bar{\zeta}_R\gamma_\nu + \frac{2i}{R}\bar{\zeta}_R\gamma^0\alpha^i\beta^j \mathrm{Tr}(X_i\dot{Y}_j - \dot{X}_iY_j) \\ &\quad - \frac{2}{R}\epsilon^{ijk}\bar{\zeta}_R\gamma^0\gamma_5(\alpha^k \mathrm{Tr}X_i\dot{X}_j + \beta^k \mathrm{Tr}Y_i\dot{Y}_j) + \cdots, \\ \delta_L \bar{J}^0_R &= -2iT^{0\nu}\bar{\zeta}_L\gamma_\nu + \frac{2i}{R^2}\bar{\zeta}_L\gamma^0 \mathrm{Tr}(X_i^2 + Y_j^2) - \frac{2}{R}\bar{\zeta}_L\gamma^0\gamma_5 \mathrm{Tr}(X_i\dot{X}_i + Y_j\dot{Y}_j) + \cdots, \\ \delta_R \bar{J}^0_L &= -2iT^{0\nu}\bar{\zeta}_R\gamma_\nu + \frac{2i}{R^2}\bar{\zeta}_R\gamma^0 \mathrm{Tr}(X_i^2 + Y_j^2) + \frac{2}{R}\bar{\zeta}_R\gamma^0\gamma_5 \mathrm{Tr}(X_i\dot{X}_i + Y_j\dot{Y}_j) + \cdots . \end{aligned} \tag{A.6}$$

Here $T^{\mu\nu}$ is the energy-momentum tensor obtained from the SYM Lagrangian (2.1). The $\cdots$ terms may involve scalars of higher degrees, such as

$$\epsilon^{ijk}\bar{\zeta}\gamma_5\{\alpha^k\beta^l \mathrm{Tr}([X_i, Y_l]X_j) + i\gamma_5\alpha^l\beta^k \mathrm{Tr}([X_l, Y_i]Y_j)\}. \tag{A.7}$$

We have checked that for the R-balls we obtained in the text, whether commutative or not, the terms presented in (A.6) are the only non-vanishing ones. For example, it is easy to verify that the terms in (A.7) vanishes for both 1/2 and 1/4 BPS R-ball solutions obtained in the text.

We introduce the following "matrix charges":

$$\begin{aligned} P_{L,R} &= \int_{\mathbf{S}^3} \gamma^0 M^\dagger_{L,R}\gamma^0\gamma_\nu M_{L,R}T^{0\nu} = P^a_{L,R}\Gamma_a + S^a_{L,R}\Gamma_a\Gamma_5, \\ K &= \int_{\mathbf{S}^3} \gamma^0 M^\dagger_L\gamma^0 \left[\gamma_\nu T^{0\nu} - \frac{1}{R^2}\gamma^0\mathrm{Tr}(X_i^2 + Y_j^2)\right] M_R, \end{aligned} \tag{A.8}$$

and the "angular momenta":

$$\begin{aligned} L_{ij} &= \frac{1}{R}\int_{\mathbf{S}^3} M^\dagger_{L,R}M_{L,R}\mathrm{Tr}(X_i\dot{Y}_j - \dot{X}_iY_j) = \frac{1}{R}\int_{\mathbf{S}^3}\mathrm{Tr}(X_i\dot{Y}_j - \dot{X}_iY_j), \\ L^k_X &= \frac{1}{R}\epsilon^{ijk}\int_{\mathbf{S}^3}\gamma_5 M^\dagger_{L,R}\gamma_5 M_{L,R}\mathrm{Tr}X_i\dot{X}_j = \frac{1}{R}\epsilon^{ijk}\int_{\mathbf{S}^3}\mathrm{Tr}X_i\dot{X}_j, \\ L^k_Y &= \frac{1}{R}\epsilon^{ijk}\int_{\mathbf{S}^3}\gamma_5 M^\dagger_{L,R}\gamma_5 M_{L,R}\mathrm{Tr}Y_i\dot{Y}_j = \frac{1}{R}\epsilon^{ijk}\int_{\mathbf{S}^3}\mathrm{Tr}Y_i\dot{Y}_j. \end{aligned} \tag{A.9}$$

Then with $\delta\mathcal{O} = -i\{\bar{\zeta}_0 Q, O\}$, the superconformal algebra can be written symbolically as

$$\begin{aligned} \{Q_L, \bar{Q}_L\} &= 2P_L + 2L_{ij}\alpha^i\beta^j\gamma^0 + 2i\gamma^0\gamma_5(\alpha^kL^k_X + \beta^kL^k_Y) + \cdots, \\ \{Q_R, \bar{Q}_R\} &= 2P_R - 2L_{ij}\alpha^i\beta^j\gamma^0 - 2i\gamma^0\gamma_5(\alpha^kL^k_X + \beta^kL^k_Y) + \cdots, \\ \{Q_L, \bar{Q}_R\} &= 2K + \cdots, \qquad\qquad \{Q_R, \bar{Q}_L\} = 2K^\dagger + \cdots. \end{aligned} \tag{A.10}$$

Here we have presented only terms that are relevant in this paper. The emergence of γ^0 in the terms involving "angular momenta" indicates that L's are not central charges. Because matrices α, β generate a rotation among indices of SU(4) R-symmetry group, these L's are actually associated with the R-charges. We may introduce fermionic charges Q and S by $Q_L = Q + S,\ Q_R = Q - S$. Then schematically the algebra (A.10) can be rewritten as

$$\begin{aligned} \{Q, \bar{Q}\} &= P_L + P_R + (K + K^\dagger) + \cdots, \qquad \{S, \bar{S}\} = P_L + P_R - (K + K^\dagger) + \cdots, \\ \{Q, \bar{S}\} &= (K - K^\dagger) + 2L + \cdots, \qquad \{S, \bar{Q}\} = -(K - K^\dagger) + 2L + \cdots. \end{aligned} \tag{A.11}$$

As the radius R of S^3 goes to infinity, we have $P_{L,R}, K, K^\dagger \rightarrow \not{p}$ with p the four momentum in four-dimensional Minkowski spacetime, M^4. Therefore, the superconformal algebra (A.11) reduces to the standard form in M^4.

The general expression for the BPS bound can be obtained by computing the eigenvalues of the right-hand side of the algebra (A.10), but the computation would be very tedious due to the presence of many off-diagonal elements. Here we only focus on the BPS bound for the supersymmetric configurations found in the text. It is not hard to see that we always have $K = 0$ for these backgrounds. For the commutative half BPS R-balls, with only one pair of scalar and pseudo-scalar turned on, we have $L_X = L_Y = 0$. Meanwhile, for our non-commutative R-balls, $L_X,\ L_Y \propto \mathrm{Tr}(a_1 \times a_2) = 0$ by using (4.18). Notice that $T^{0i} = 0$ for these configurations. Computing the eigenvalues on the right side of the superconformal algebra (A.10), we obtain the BPS bound for energy:

$$E \geq |L|, \tag{A.12}$$

where the angular momentum L is nothing but just R-charge $Q_{\mathbf{r}}$ defined in eq, (3.3). For commutative half BPS R-balls, this BPS bound is exactly saturated, while for the non-commutative 1/4 BPS R-balls, their energy receives an extra contribution from the tree-level quartic interactions, which is of order $\lambda/N(N_1 + N_2)$ and can be ignored in the limit $N \to \infty$.

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